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# A NOTE ON GENERALISED LORENTZIAN SASAKIAN MANIFOLD

**L K Pandey**
D S Institute of Technology & Management, Ghaziabad (U.P.) 201007, India

## ABSTRACT
In 2011, R. Nivas and A. Bajpai [2] discussed on generalized Lorentzian Para-Sasakian manifolds. In 1994, L.K.Pandey and R.H.Ojha [3] studied on LP-contact manifold. In 1970, K. Yano [5] discussed on semi symmetric metric connections. Symmetric metric connections are also discussed by Nirmala S. Agashe and Mangala R. Chafle [1], R. N. Singh and S. K. Pandey [4] and many others. In this paper, generalised nearly Lorentzian Sasakian and generalised almost Lorentzian Sasakian manifolds have been discussed and some of their properties have been established with generalised Lorentzian Co-symplectic manifolds. Semi-symmetric metric F-connection in a generalised Lorentzian Sasakian manifold has also been discussed.

**KEYWORDS**: Generalised nearly Lorentzian Sasakian manifold, generalised almost Lorentzian Sasakian manifold, generalised Lorentzian Co-symplectic manifolds, semi-symmetric metric F-connection.

## INTRODUCTION
An $n\,(=2m+1)$ dimensional differentiable manifold $M_n$, which admits a tensor field $F$of type (1, 1), two contravariant vector fields $T_1$and $T_2$, two covariant vector fields $A_1$and $A_2$ and a Lorentzian metric $g$, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\overline{\overline{X}} = -X - A_1(X)T_1 - A_2(X)T_2,\ \overline{T_1} = 0,\ \overline{T_2} = 0,\ A_1(T_1) = -1,\ A_2(T_2) = -1,\ \overline{X} \stackrel{\text{def}}{=} FX,\ A_1(\overline{X}) = 0,$

$A_2(\overline{X}) = 0$, rank $F = n-2$

(1.2) $g\,(\overline{X}, \overline{Y}) = g\,(X, Y) + A_1(X)A_1(Y) + A_2(X)A_2(Y)$, where $A_1(X) = g(X, T_1),\ A_2(X) = g(X, T_2)$

$`F(X, Y) \stackrel{\text{def}}{=} g\left(\overline{X},\ Y\right)$

Then $M_n$ is called a generalised Lorentzian contact manifold ( generalised L-contact manifold ) and the structure $(F, T_1, T_2, A_1, A_2, g)$ is known as generalised Lorentzian contact structure
On a generalised L-contact manifold, we have

(1.3) (a) $`F(X, Y) + `F(Y, X) = 0$  (b) $`F\left(\overline{X},\ \overline{Y}\right) = `F(X, Y)$

(c) $(D_X`F)(Y, T_1) = -(D_XA_1)\left(\overline{Y}\right)$  (d) $(D_X`F)(Y, T_2) = -(D_XA_2)\left(\overline{Y}\right)$

(1.4) (a) $(D_X`F)\left(\overline{Y},\ Z\right) - (D_X`F)\left(Y, \overline{Z}\right) + A_1(Y)(D_XA_1)(Z) + A_2(Y)(D_XA_2)(Z) + A_1(Z)(D_XA_1)(Y) +$

$A_2(Z)(D_XA_2)(Y) = 0$

(b) $(D_X`F)\left(\overline{Y},\ \overline{\overline{Z}}\right) = (D_X`F)\left(\overline{\overline{Y}}, \overline{Z}\right)$

(c) $(D_X`F)\left(\overline{Y},\ \overline{Z}\right) + (D_X`F)(Y, Z) + A_1(Y)(D_XA_1)\left(\overline{Z}\right) + A_2(Y)(D_XA_2)\left(\overline{Z}\right) - A_1(Z)(D_XA_1)\left(\overline{Y}\right) -$

$A_2(Z)(D_XA_2)\left(\overline{Y}\right) = 0$

(d) $(D_X`F)\left(\overline{\overline{Y}}, \overline{\overline{Z}}\right) + (D_X`F)\left(\overline{Y},\ \overline{Z}\right) = 0$

Where D is the Riemannian connection on $M_n$.

A generalised L-contact manifold is called a generalised Lorentzian Sasakian manifold if

(1.5) (a) $2(D_XF)(Y) - \{A_1(Y) + A_2(Y)\}\overline{\overline{X}} - g(\overline{X},\overline{Y})(T_1 + T_2) = 0 \Leftrightarrow$

(b) $2(D_X`F)(Y,Z) + \{A_1(Y) + A_2(Y)\}\, g(\overline{X},\overline{Z}) - \{A_1(Z) + A_2(Z)\}g(\overline{X},\overline{Y}) = 0 \Leftrightarrow$

(c) $2D_XT_1 = \overline{X} - T_2, \qquad 2\, D_XT_2 = \overline{X} - T_1$

From which, we get

(1.6) (a) $2(D_XA_1)(\overline{Y}) = 2(D_XA_2)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow$

(b) $2(D_XA_1)(Y) + A_2(Y) = 2(D_XA_2)(Y) + A_1(Y) = `F(X,Y) \Leftrightarrow$

And

(1.7) (a) $2(D_X`F)(\overline{Y},Z) + \{A_1(Z) + A_2(Z)\}`F(X,Y) = 0$

(b) $2(D_X`F)\left(\overline{\overline{Y}},Z\right) + \{A_1(Z) + A_2(Z)\}g(\overline{X},\overline{Y}) = 0$

(c) $(D_X`F)(Y,Z) + A_1(Y)(D_XA_1)(\overline{Z}) + A_2(Y)(D_XA_2)(\overline{Z}) - A_1(Z)(D_XA_1)(\overline{Y}) - A_2(Z)(D_XA_2)(\overline{Z}) = 0$

Nijenhuis tensor in a generalised L-contact manifold is given as

(1.8) $`N(X,Y,Z) = (D_{\overline{X}}`F)(Y,Z) + (D_{\overline{Y}}`F)(Z,X) + (D_X`F)(Y,\overline{Z}) + (D_Y`F)(\overline{Z},X)$

Where $`N(X,Y,Z) \overset{\text{def}}{=} g(N(X,Y),Z)$

## GENERALISED NEARLY AND ALMOST LORENTZIAN SASAKIAN MANIFOLDS

A generalised L-contact manifold will be called a generalised nearly Lorentzian Sasakian manifold if

(2.1) $2(D_X`F)(Y,Z) + \{A_1(Y) + A_2(Y)\}\, g(\overline{X},\overline{Z}) - \{A_1(Z) + A_2(Z)\}\, g(\overline{X},\overline{Y})$

$= 2(D_Y`F)(Z,X) + \{A_1(Z) + A_2(Z)\}\, g(\overline{X},\overline{Y}) - \{A_1(X) + A_2(X)\}\, g(\overline{Y},\overline{Z})$

$= 2(D_Z`F)(X,Y) + \{A_1(X) + A_2(X)\}\, g(\overline{Y},\overline{Z}) - \{A_1(Y) + A_2(Y)\}\, g(\overline{X},\overline{Z})$

The equation of a generalised nearly Lorentzian Sasakian manifold can be modified as

(2.2) (a) $2(D_XF)Y + 2(D_YF)X - \{A_1(Y) + A_2(Y)\}\overline{\overline{X}} - \{A_1(X) + A_2(X)\}\overline{\overline{Y}} - 2g(\overline{X},\overline{Y})(T_1 + T_2) = 0 \Leftrightarrow$

(b) $2(D_X`F)(Y,Z) + 2(D_Y`F)(X,Z) + \{A_1(Y) + A_2(Y)\}\, g(\overline{X},\overline{Z}) + \{A_1(X) + A_2(X)\}\, g(\overline{Y},\overline{Z}) -$
$2\{A_1(Z) + A_2(Z)\}\, g(\overline{X},\overline{Y}) = 0$

This gives

(2.3) (a) $2(D_XF)\overline{Y} + 2(D_{\overline{Y}}F)X + \{A_1(X) + A_2(X)\}\overline{Y} + 2`F(X,Y)(T_1 + T_2) = 0 \Leftrightarrow$

(b) $2(D_X`F)(\overline{Y},Z) - 2(D_{\overline{Y}}`F)(Z,X) + \{A_1(X) + A_2(X)\}\, `F(Y,Z) + 2\{A_1(Z) + A_2(Z)\}`F(X,Y) = 0$

(2.4) (a) $2(D_XF)\overline{\overline{Y}} + 2(D_{\overline{\overline{Y}}}F)X + \{A_1(X) + A_2(X)\}\overline{\overline{Y}} + 2g(\overline{X},\overline{Y})(T_1 + T_2) = 0 \Leftrightarrow$

(b) $2(D_X`F)\left(\overline{\overline{Y}},Z\right)-2\left(D_{\overline{\overline{Y}}}`F\right)(Z,X)-\{A_1(X)+A_2(X)\}g\left(\overline{Y},\overline{Z}\right)+2\{A_1(Z)+A_2(Z)\}\,g\left(\overline{X},\overline{Y}\right)=0$

(2.5) (a) $(D_XF)Y+(D_YF)X-A_1(Y)\{\overline{D_XT_1}-(D_{T_1}F)X\}-A_2(Y)\{\overline{D_XT_2}-(D_{T_2}F)X\}-A_1(X)\{\overline{D_YT_1}-(D_{T_1}F)Y\}-A_2(X)\{\overline{D_YT_2}-(D_{T_2}F)Y\}-g\left(\overline{X},\overline{Y}\right)(T_1+T_2)=0\Leftrightarrow$

(b) $(D_X`F)(Y,Z)+(D_Y`F)(X,Z)+A_1(Y)\{(D_XA_1)\left(\overline{Z}\right)-(D_{T_1}`F)(Z,X)\}+A_2(Y)\{(D_XA_2)\left(\overline{Z}\right)-(D_{T_2}`F)(Z,X)\}+A_1(X)\{(D_YA_1)\left(\overline{Z}\right)+(D_{T_1}`F)(Y,Z)\}+A_2(X)\{(D_YA_2)\left(\overline{Z}\right)+(D_{T_2}`F)(Y,Z)\}-\{A_1(Z)+A_2(Z)\}\,g\left(\overline{X},\overline{Y}\right)=0$

A generalised L-contact manifold will be called a generalised almost Lorentzian Sasakian manifold if

(2.6) $(D_X`F)(Y,Z)+(D_Y`F)(Z,X)+(D_Z`F)(X,Y)=0$

## GENERALISED LORENTZIAN CO_SYMPLECTIC MANIFOLDS

A generalised L-contact manifold will be called a generalised Lorentzian Co-symplectic manifold if

(3.1) (a) $(D_XF)Y-A_1(Y)\overline{D_XT_1}-A_2(Y)\overline{D_XT_2}-(D_XA_1)\left(\overline{Y}\right)T_1-(D_XA_2)\left(\overline{Y}\right)T_2=0\Leftrightarrow$

(b) $(D_X`F)(Y,Z)+A_1(Y)(D_XA_1)\left(\overline{Z}\right)+A_2(Y)(D_XA_2)\left(\overline{Z}\right)-A_1(Z)(D_XA_1)\left(\overline{Y}\right)-A_2(Z)(D_XA_2)\left(\overline{Y}\right)=0$

Therefore a generalised Lorentzian Co-symplectic manifold is a generalised Lorentzian Sasakian manifold if

(3.2) (a) $2(D_XA_1)\left(\overline{Y}\right)=2(D_XA_2)\left(\overline{Y}\right)=g\left(\overline{X},\ \overline{Y}\right)\Leftrightarrow$

(b) $2(D_XA_1)(Y)+A_2(Y)=2(D_XA_2)(Y)+A_1(Y)=`F(X,Y)\Leftrightarrow$ (c) $2D_XT_1=\overline{X}-T_2,\quad 2D_XT_2=\overline{X}-T_1$

A generalised L-contact manifold will be called a generalised nearly Lorentzian Co-symplectic manifold if

(3.3) $(D_X`F)(Y,Z)+A_1(Y)(D_XA_1)\left(\overline{Z}\right)+A_2(Y)(D_XA_2)\left(\overline{Z}\right)-A_1(Z)(D_XA_1)\left(\overline{Y}\right)-A_2(Z)(D_XA_2)\left(\overline{Y}\right)$

$=(D_Y`F)(Z,X)+A_1(Z)(D_YA_1)\left(\overline{X}\right)+A_2(Z)(D_YA_2)\left(\overline{X}\right)-A_1(X)(D_YA_1)\left(\overline{Z}\right)-A_2(X)(D_YA_2)\left(\overline{Z}\right)$

$=(D_Z`F)(X,Y)+A_1(X)(D_ZA_1)\left(\overline{Y}\right)+A_2(X)(D_ZA_2)\left(\overline{Y}\right)-A_1(Y)(D_ZA_1)\left(\overline{X}\right)-A_2(Y)(D_ZA_2)\left(\overline{X}\right)$

It is clear that a generalised nearly Lorentzian Sasakian manifold is a generalised nearly Lorentzian Co-symplectic manifold, in which

(3.4) (a) $2(D_XA_1)\left(\overline{Y}\right)=2(D_XA_2)\left(\overline{Y}\right)=g\left(\overline{X},\ \overline{Y}\right)\Leftrightarrow$

(b) $2(D_XA_1)(Y)+A_2(Y)=2(D_XA_2)(Y)+A_1(Y)=`F(X,Y)\Leftrightarrow$ (c) $2D_XT_1=\overline{X}-T_2,\quad 2D_XT_2=\overline{X}-T_1$

A generalised L-contact manifold will be called a generalised almost Lorentzian Co-symplectic manifold if

(3.5) $(D_X`F)(Y,Z)+(D_Y`F)(Z,X)+(D_Z`F)(X,Y)-A_1(X)\{(D_YA_1)\left(\overline{Z}\right)-(D_ZA_1)\left(\overline{Y}\right)\}-A_2(X)\{(D_YA_2)\left(\overline{Z}\right)-(D_ZA_2)\left(\overline{Y}\right)\}-A_1(Y)\{(D_ZA_1)\left(\overline{X}\right)-(D_XA_1)\left(\overline{Z}\right)\}-A_2(Y)\{(D_ZA_2)\left(\overline{X}\right)-(D_XA_2)\left(\overline{Z}\right)\}-A_1(Z)\{(D_XA_1)\left(\overline{Y}\right)-(D_YA_1)\left(\overline{X}\right)\}-A_2(Z)\{(D_XA_2)\left(\overline{Y}\right)-(D_YA_2)\left(\overline{X}\right)\}=0$

Therefore, A generalised almost Lorentzian Co-symplectic manifold is a generalised almost Lorentzian Sasakian manifold if

(3.6) (a) $2(D_XA_1)\left(\overline{Y}\right)=2(D_XA_2)\left(\overline{Y}\right)=g\left(\overline{X},\ \overline{Y}\right)\Leftrightarrow$

(b) $2(D_X A_1)(Y) + A_2(Y) = 2(D_X A_2)(Y) + A_1(Y) = `F(X,Y) \Leftrightarrow$ (c) $2D_X T_1 = \overline{X} - T_2, \quad 2D_X T_2 = \overline{X} - T_1$

## COMPLETELY INTEGRABLE MANIFOLDS

Barring $X, Y, Z$ in (1.8) and using equations (2.1), (1.4) (b), we see that a generalised nearly Lorentzian Sasakian manifold is completely integrable if

(4.1) $(D_{\overline{X}} `F)\left(\overline{Y}, \overline{\overline{Z}}\right) = (D_{\overline{Y}} `F)(\overline{X}, \overline{\overline{Z}})$

Barring X, Y, Z in (1.8) and using equations (2.6), (1.4) (b), we see that a generalised almost Lorentzian Sasakian manifold is completely integrable if

(4.2) $(D_{\overline{Z}} `F)(\overline{X}, \overline{\overline{Y}}) = 0.$

## SEMI_SYMMETRIC METRIC F-CONNECTION IN A GENERALISED LORENTZIAN SASAKIAN MANIFOLD

Let $M_{2m-1}$ be submanifold of $M_{2m+1}$ and let $c : M_{2m-1} \to M_{2m+1}$ be the inclusion map such that

$$d \in M_{2m-1} \to cd \in M_{2m+1} ,$$

Where $c$ induces a linear transformation (Jacobian map) $J : T'_{2m-1} \to T'_{2m+1}$.

$T'_{2m-1}$ is a tangent space to $M_{2m-1}$ at point $d$ and $T'_{2m+1}$ is a tangent space to $M_{2m+1}$ at point $cd$ such that

$$\hat{X} \text{ in } M_{2m-1} \text{ at } d \to J\hat{X} \text{ in } M_{2m+1} \text{ at } cd$$

Let $\tilde{g}$ be the induced Lorentzian metric in $M_{2m-1}$ . Then we have

(5.1) $\tilde{g}(\hat{X}, \hat{Y}) = ((g(J\hat{X}, J\hat{Y}))b$

We now suppose that a semi-symmetric metric F-connection $B$ in a generalised Lorentzian Sasakian manifold is given by

(5.2) $2B_X Y = 2D_X Y + A_1(Y)FX + A_2(Y)FX - g(FX,Y)T_1 - g(FX,Y)T_2 + 2\{A_1(X) + A_2(X)\}FY ,$

Where $X$ and $Y$ are arbitrary vector fields of $M_{2m+1}$. If

(5.3) (a) $T_1 = Jt_1 + \rho_1 M + \sigma_1 N$ and

(b) $T_2 = Jt_2 + \rho_2 M + \sigma_2 N$

Where $t_1$ and $t_2$ are $C^\infty$ vector fields in $M_{2m-1}$ and $M$ and $N$ are unit normal vectors to $M_{2m-1}$.

Denoting by $\grave{D}$ the connection induced on the submanifold from $D$, we have Gauss equation

(5.4) $D_{JX} J\hat{Y} = J(\grave{D}_X \hat{Y}) + h(\hat{X}, \hat{Y})M + k(\hat{X}, \hat{Y})N$

Where $h$ and $k$ are symmetric bilinear functions in $M_{2m-1}$. Similarly we have

(5.5) $B_{JX} J\hat{Y} = J(\grave{B}_X \hat{Y}) + p(\hat{X}, \hat{Y})M + q(\hat{X}, \hat{Y})N ,$

Where $\grave{B}$ is the connection induced on the submanifold from $B$ and $p$ and $q$ are symmetric bilinear functions in $M_{2m-1}$

Inconsequence of (5.2), we have

(5.6) $2B_{JX}J\hat{Y} = 2D_{JX}J\hat{Y} + A_1(J\hat{Y})JF\hat{X} + A_2(J\hat{Y})JF\hat{X} - g(JF\hat{X}, J\hat{Y})T_1 - g(JF\hat{X}, J\hat{Y})T_2 + 2\{A_1(J\hat{X})JF\hat{Y} + A_2(J\hat{X})JF\hat{Y}\}$

Using (5.4), (5.5) and (5.6), we get

(5.7) $2\,J(\grave{B}_X\hat{Y}) + 2p(\hat{X},\hat{Y})M + 2q(\hat{X},\hat{Y})N = 2J(\grave{D}_X\hat{Y}) + 2h(\hat{X},\hat{Y})M + 2k(\hat{X},\hat{Y})N + A_1(J\hat{Y})JF\hat{X} + A_2(J\hat{Y})JF\hat{X} - g(JF\hat{X}, J\hat{Y})T_1 - g(JF\hat{X}, J\hat{Y})T_2 + 2\{A_1(J\hat{X})JF\hat{Y} + A_2(J\hat{X})JF\hat{Y}\}$

Using (5.3) (a) and (5.3) (b), we obtain

(5.8) $2J(\grave{B}_X\hat{Y}) + 2p(\hat{X},\hat{Y})M + 2q(\hat{X},\hat{Y})N = 2J(\grave{D}_X\hat{Y}) + 2h(\hat{X},\hat{Y})M + 2k(\hat{X},\hat{Y})N + a_1(\hat{Y})JF\hat{X} + a_2(\hat{Y})JF\hat{X} - \tilde{g}(F\hat{X},\hat{Y})(Jt_1 + \rho_1 M + \sigma_1 N) - \tilde{g}(F\hat{X},\hat{Y})(Jt_2 + \rho_2 M + \sigma_2 N) + 2\{a_1(\hat{X})JF\hat{Y} + a_2(\hat{X})JF\hat{Y}\}$

Where $\tilde{g}(\hat{Y}, t_1) \overset{\text{def}}{=} a_1(\hat{Y})$ and $\tilde{g}(\hat{Y}, t_2) \overset{\text{def}}{=} a_2(\hat{Y})$

Which implies

(5.9) $2\grave{B}_X\hat{Y} = 2\grave{D}_X\hat{Y} + a_1(\hat{Y})F\hat{X} + a_2(\hat{Y})F\hat{X} - \tilde{g}(F\hat{X},\hat{Y})t_1 - \tilde{g}(F\hat{X},\hat{Y})t_2 + 2\{a_1(\hat{X})F\hat{Y} + a_2(\hat{X})F\hat{Y}\}$

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(5.10) (a) $2p(\hat{X},\hat{Y}) = 2h(\hat{X},\hat{Y}) - \rho_1\tilde{g}(F\hat{X},\hat{Y}) - \rho_2\tilde{g}(F\hat{X},\hat{Y})$

(b) $2q(\hat{X},\hat{Y}) = 2k(\hat{X},\hat{Y}) - \sigma_1\tilde{g}(F\hat{X},\hat{Y}) - \sigma_2\tilde{g}(F\hat{X},\hat{Y})$

Thus we have

**Theorem 5.1** The connection induced on a submanifold of a generalised Lorentzian Sasakian manifold with a semi-symmetric metric F-connection with respect to unit normal vectors $M$ and $N$ is also semi-symmetric metric F-connection iff (5.10) holds.

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